

New Integral Inequalities of Hadamard's for Harmonically Convex Stochastic Processes on n-coordinates

Nurgül Okur¹, Vildan Karahan²

¹ Giresun University, Department of Statistics, Giresun/Turkey and
nrgokur@gmail.com

² Institute of Sciences, Giresun University, 28200 Giresun, Turkey.
vildan.karahan2@ugurokullari.k12.tr

ABSTRACT. In this study, we consider harmonically convex stochastic processes on n-coordinates. Also, we obtain Hermite-Hadamard type integral inequalities for these processes on n-coordinates.

Keywords: Harmonically convexity, stochastic process on n-coordinates, mean-square integral, Hermite-Hadamard inequality.

AMS Classification 2010: 37A50, 32F17.

1. INTRODUCTION

It is well known that, for every real convex function f on the interval $[a, b]$, we have [1]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

These are celebrated Hermite-Hadamard inequalities. In probabilistic words, they say that

$$f(EX) \leq_{cx} Ef(X) \leq_{cx} Ef(X^*), f \in C_{cx}$$

where E denotes mathematical expectation, X (respectively, X^*) is a random variable having the uniform distribution on the interval $[a, b]$ (respectively, on the set $\{a, b\}$), C_{cx} is the set of all real convex functions on $[a, b]$ and \leq_{cx} stands for the so called convex order of random variables [2].

Alomari [3] generalized the classical Hermite-Hadamard type inequalities for every convex function and for every positive convex function on $[a, b]$. Mwaeze [4] obtained some generalized Hermite-Hadamard's inequality for every convex function and for every positive convex function on the coordinates. Elahi [5] obtained Hadamard's inequality for s-convex function on n-coordinates.

There are many studies in recent years on some types of convexity for stochastic processes and Hermite-Hadamard inequalities for related convex stochastic processes, and it is of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities in the literature. Convex stochastic processes were proposed and some properties were given for classical convex stochastic processes by Nikodem [6]. Stochastic convexity and its applications were defined by Shaked et al [7]. Jensen-convex, λ -convex stochastic processes were introduced by Skowronski [8]. The classical Hermite-Hadamard inequality to convex stochastic processes was extended by

Kotrys [9]. Convex stochastic processes on the coordinates were introduced and Hermite-Hadamard type inequalities for these processes were obtained by Set et al [10]. Harmonically convex stochastic processes were defined and Hermite-Hadamard type inequalities were obtained by Okur et al [11]. Okur et al [12] extended these processes on two-dimensional and interval obtained Hermite-Hadamard type inequalities and estimation for these processes. Karahan et al [13] investigated the convex stochastic processes on n-coordinates, and obtained Hermite-Hadamard type inequality for these processes.

The authors' findings led to our motivation to build our work. The main subject of this paper is to adapt some obtained results concerning harmonically convex functions on n-coordinates by Vitoria et al [14] to harmonically convex stochastic processes on n-coordinates and to obtain Hermite-Hadamard type inequalities.

2. PRELIMINARIES

There are well-known definitions of stochastic processes and some fundamentals about Hermite Hadamard Inequality for stochastic processes in the literature (see, [9]-[13]).

Definition 2.1[9]. Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space and $I \in \mathfrak{F}$, $I \subset \mathbb{R}$ be an interval. The stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ is called almost everywhere convex if

$$X(\lambda t + (1 - \lambda)s, \cdot) \leq \lambda X(t, \cdot) + (1 - \lambda)X(s, \cdot)$$

for all $t, s \in I$ and $\lambda \in [0, 1]$. If the above inequality is reversed then X is said to be concave.

Definition 2.2[9]. Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space and $I \in \mathfrak{F}$, $I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ is called

(i) continuous in probability on I if for all $t_0 \in I$ if

$$P - \lim_{t \rightarrow t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes limit in probability,

(ii) mean-square continuous on I if for all $t_0 \in I$ if

$$\lim_{t \rightarrow t_0} E[X(t, \cdot) - X(t_0, \cdot)]^2 = 0,$$

where $E[\xi(t, \cdot)]$ denotes expectation value of the random variable $X(t, \cdot)$,

(iii) mean-square differentiable at a point if $t \in I$ if there is a random variable $X'(t, \cdot): I \times \Omega \rightarrow \mathbb{R}$ such that

$$X'(t, \cdot) = P - \lim_{t \rightarrow t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

We say that a stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I .

Definition 2.3[9]. Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space and $I \in \mathfrak{F}$, $I \subset \mathbb{R}$ be an interval and $X: I \times \Omega \rightarrow \mathbb{R}$ be a stochastic process with $E[X(t, \cdot)^2] \leq \infty$ for all $t \in I$. Let

$[0, t] \subset I$, $0 = t_0 < t_1 \dots t_n = t$ be a partition of $[0, t]$ and $\theta_k \in [t_{k-1}, t_k]$ arbitrary for $k = 1, \dots, n$. A random variable $\eta: \Omega \times \Omega \rightarrow \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[0, t]$ if the following identity holds:

$$\lim_{n \rightarrow \infty} E \left(\sum_{k=1}^n X(\theta_k, \cdot) \cdot (t_k - t_{k-1}) - \eta(t, \cdot) \right)^2 = 0.$$

Then we can write almost everywhere

$$\int_0^t X(u, \cdot) du = \eta(t, \cdot).$$

The mean-square integral operator is increasing on $[0, t]$ almost everywhere, that is,

$$X(t, \cdot) \leq Y(t, \cdot) \Rightarrow \int_0^t X(t, \cdot) dt \leq \int_0^t Y(t, \cdot) dt.$$

Let us consider the Hermite-Hadamard integral inequality for harmonically convex stochastic processes:

Definition 2.4[11]. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ is said to be a harmonically convex stochastic process almost everywhere, if

$$X\left(\frac{ts}{\lambda t + (1-\lambda)s}, \cdot\right) \leq \lambda X(s, \cdot) + (1-\lambda)X(t, \cdot) \quad (2.1)$$

for all $t, s \in I$ and $\lambda \in [0, 1]$. If the above inequality is reversed, then X is said to be a harmonically concave almost everywhere.

Proposition 2.1[11]. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $X: I \times \Omega \rightarrow \mathbb{R}$ is a stochastic process.

(i) If X is convex and nondecreasing stochastic process, then X is harmonically convex.

(ii) If X is harmonically convex and nonincreasing stochastic process, then X is convex.

Theorem 2.3[11]. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $X: I \times \Omega \rightarrow \mathbb{R}$ be a harmonically convex stochastic process and $a, b \in I^\circ$ with $a < b$. If $X \in L[a, b]$ then the following inequalities hold almost everywhere

$$X\left(\frac{2ab}{a+b}, \cdot\right) \leq \frac{ab}{b-a} \int_a^b \frac{X(t, \cdot)}{t^2} dt \leq \frac{X(a, \cdot) + X(b, \cdot)}{2}. \quad (2.2)$$

Let us consider the two-dimensional interval $\Delta = [a, b] \times [c, d]$ in $(0, \infty) \times (0, \infty)$ with $a < b$ and $c < d$.

Definition 2.5[12]. A stochastic process $X: \Delta \times \Omega \rightarrow \mathbb{R}$ is said to be a harmonically convex on Δ , if the following inequality holds almost everywhere

$$X\left(\left(\frac{t_1 t_2}{\lambda t_1 + (1-\lambda)t_2}, \frac{s_1 s_2}{\lambda s_1 + (1-\lambda)s_2}\right), \cdot\right) \leq \lambda X((t_2, s_2), \cdot) + (1-\lambda)X((t_1, s_1), \cdot)$$

for all $(t_1, s_1), (t_2, s_2) \in \Delta$ and $\lambda \in [0,1]$. If the above inequality is reversed, then X is said to be a harmonically concave on Δ .

Definition 2.6[12]. A stochastic process $X: \Delta \times \Omega \rightarrow \mathbb{R}$ is said to be a harmonically convex on the co-ordinates on $\Delta \times \Omega$ if the partial mappings $X_s: [a, b] \times \Omega \rightarrow \mathbb{R}, X_s(u, \cdot) := X((u, s), \cdot)$ and $X_t: [c, d] \times \Omega \rightarrow \mathbb{R}, X_t(v, \cdot) := X((t, v), \cdot)$ defined for all $t \in [a, b]$ and $s \in [c, d]$ are harmonically convex almost everywhere.

Theorem 2.1[12]. Suppose that $X: \Delta \times \Omega \rightarrow \mathbb{R}$ is harmonically convex on the coordinates on Δ . Then the following inequalities hold almost everywhere:

$$\begin{aligned} & X\left(\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right), \cdot\right) \\ & \leq \frac{1}{2} \left[\frac{ab}{b-a} \int_a^b \frac{1}{t^2} X\left(\left(t, \frac{2cd}{c+d}\right), \cdot\right) dt + \frac{cd}{d-c} \int_c^d \frac{1}{s^2} X\left(\left(\frac{2ab}{a+b}, s\right), \cdot\right) ds \right] \\ & \leq \frac{abcd}{(b-a)(d-c)} \int_a^b \int_c^d \frac{1}{(ts)^2} X((t, s), \cdot) dt ds \\ & \leq \frac{1}{4} \left[\frac{ab}{b-a} \int_a^b \frac{1}{t^2} \left(X((t, c), \cdot) + X((t, d), \cdot) \right) dt + \frac{cd}{d-c} \int_c^d \frac{1}{s^2} \left(X((a, s), \cdot) + X((b, s), \cdot) \right) ds \right] \\ & \leq \frac{X((a, c), \cdot) + X((b, c), \cdot) + X((a, d), \cdot) + X((b, d), \cdot)}{4} \quad (2.3) \end{aligned}$$

For $n \geq 2$, let $u_i, v_i, (i = 1, 2, \dots, n)$ be real numbers such that $u_i < v_i$ for $i = 1, 2, \dots, n$, and the n -dimensional interval $\Delta^n = \prod_{i=1}^n [u_i, v_i] \subseteq [0, \infty)^n$.

In the following we give definition of convexity for stochastic processes on n -coordinates:

Definition 2.7[13]. A stochastic process $X: \Delta^n \times \Omega \rightarrow \mathbb{R}$ is called convex on n -coordinates if the stochastic processes

$$X_{t_n}^i(t, \cdot) := X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot)$$

are convex on $[u_i, v_i]$ almost everywhere for $i = 1, 2, \dots, n$.

Definition 2.8[13]. A stochastic process $X: \Delta^n \times \Omega \rightarrow \mathbb{R}$ is said to be convex on Δ^n if the following inequality holds almost everywhere

$$X((\lambda \mathbf{t} + (1-\lambda)\mathbf{s}), \cdot) \leq \lambda X(\mathbf{t}, \cdot) + (1-\lambda)X(\mathbf{s}, \cdot)$$

for all $\mathbf{t} = (t_1, t_2, \dots, t_n), \mathbf{s} = (s_1, s_2, \dots, s_n) \in \Delta^n$ and $\lambda \in [0,1]$. If the above inequality is reversed then X is said to be concave on Δ^n .

3. MAIN RESULTS

The main goal of this section is to present Hermite-Hadamard type inequalities for harmonically convex stochastic processes on n -coordinates.

In the following we give definition of harmonically convexity for stochastic processes on n -coordinates:

Definition 3.1. A stochastic process $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ is called convex on n -coordinates if the stochastic processes

$$X_{t_n}^i(t, \cdot) := X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot)$$

are harmonically convex on $[u_i, v_i]$ almost everywhere for $i = 1, 2, \dots, n$.

Definition 3.2. A stochastic process $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ is said to be harmonically convex on Δ^n if the following inequality holds almost everywhere

$$X\left(\left(\frac{ts}{\lambda t + (1-\lambda)s}\right), \cdot\right) \leq \lambda X(s, \cdot) + (1-\lambda)X(t, \cdot) \quad (3.1)$$

for all $\mathbf{t} = (t_1, t_2, \dots, t_n), \mathbf{s} = (s_1, s_2, \dots, s_n) \in \Delta^n$ and $\lambda \in [0, 1]$. If the above inequality is reversed then X is said to be harmonically concave on Δ^n .

Proposition 3.1. Let be $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ is a stochastic process and $X_{t_n}^i(t, \cdot) := X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot)$, for some $i = 1, 2, \dots, n$.

(i) If $X_{t_n}^i$ is convex and nondecreasing stochastic process for $i = 1, 2, \dots, n$, then X is harmonically convex on n -coordinates.

(ii) If $X_{t_n}^i$ is harmonically convex and nonincreasing stochastic process for $i = 1, 2, \dots, n$, then X is convex on n -coordinates.

Lemma 3.1. Every harmonically convex stochastic process $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ is harmonically convex on n -coordinates almost everywhere but converse is not true.

Proof. Let $X: \Delta^n \times \Omega \rightarrow \mathbb{R}$ be harmonically convex on Δ^n . Assume that $X_{t_n}^i: [u_i, v_i] \times \Omega \rightarrow \mathbb{R}$, defined by

$$X_{t_n}^i(t, \cdot) := X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot), \quad t \in [u_i, v_i].$$

Now for $t, s \in [u_i, v_i]$ and $\lambda \in [0, 1]$ almost everywhere

$$\begin{aligned} X_{t_n}^i\left(\left(\frac{ts}{\lambda t + (1-\lambda)s}\right), \cdot\right) &= X\left(\left(t_1, \dots, t_{i-1}, \frac{ts}{\lambda t + (1-\lambda)s}, t_{i+1}, \dots, t_n\right), \cdot\right) \\ &= X\left(\left(\left(\frac{t_1^2}{\lambda t_1 + (1-\lambda)t_1}, \dots, \frac{t_{i-1}^2}{\lambda t_{i-1} + (1-\lambda)t_{i-1}}, \frac{ts}{\lambda t + (1-\lambda)s}, \frac{t_{i+1}^2}{\lambda t_{i+1} + (1-\lambda)t_{i+1}}, \dots, \frac{t_n^2}{\lambda t_n + (1-\lambda)t_n}\right), \cdot\right)\right) \end{aligned}$$

$$\begin{aligned}
&\leq \lambda X((t_1, \dots, t_{i-1}, s, t_{i+1}, \dots, t_n), \cdot) + (1 - \lambda) X((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n), \cdot) \\
&= \lambda X_{t_n}^i(s, \cdot) + (1 - \lambda) X_{t_n}^i(t, \cdot)
\end{aligned}$$

which implies $X_{t_n}^i$ is harmonically convex on $[u_i, v_i]$, that is, X is harmonically convex on n -coordinates.

Now, consider $X: [3,6] \times [4,6] \times [5,6] \times \Omega \rightarrow [0, \infty)$ given $X((t_1, t_2, t_3), \cdot) = (t_1 - 3)(t_1 - 4)(t_3 - 5)$. It is obvious that X harmonically convex on 3-coordinates by Proposition 3.1, but is not harmonically convex on $[3,6] \times [4,6] \times [5,6]$. Indeed $(3,6,6), (4,6,5) \in [3,6] \times [4,6] \times [5,6]$ and $\lambda \in (0,1)$ we have

$$\begin{aligned}
&X\left(\left(\frac{3.4}{3.\lambda + (1-\lambda).4}, \frac{6.6}{6.\lambda + (1-\lambda).6}, \frac{6.5}{6.\lambda + (1-\lambda).5}\right), \cdot\right) \\
&= X\left(\left(\frac{12}{4-\lambda}, 6, \frac{30}{5+\lambda}\right), \cdot\right) = \left(\frac{12}{4-\lambda} - 3\right)(6-4)\left(\frac{30}{5+\lambda} - 5\right) \\
&= \left(\frac{3\lambda}{4-\lambda}\right) 2 \left(\frac{5-\lambda}{5+\lambda}\right) = \frac{6\lambda(5-\lambda)}{(4-\lambda)(5+\lambda)} > 0;
\end{aligned}$$

and

$$\begin{aligned}
&\lambda X((3,6,6), \cdot) + (1 - \lambda) X((4,6,5), \cdot) \\
&= \lambda(3-3)(6-4)(6-5) + (1 - \lambda)(4-3)(6-4)(5-5) = 0.
\end{aligned}$$

Thus for all $\lambda \in (0,1)$, we have

$$\begin{aligned}
&X\left(\left(\frac{3.4}{3.\lambda + (1-\lambda).4}, \frac{6.6}{6.\lambda + (1-\lambda).6}, \frac{6.5}{6.\lambda + (1-\lambda).5}\right), \cdot\right) \\
&> \lambda X((3,6,6), \cdot) + (1 - \lambda) X((4,6,5), \cdot)
\end{aligned}$$

which shows that X is not harmonically convex on $[3,6] \times [4,6] \times [5,6]$.

Theorem 3.1. *If $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ is harmonically convex stochastic process on n -coordinates, then $X_{t_n}^k: [u_k, v_k] \times \Omega \rightarrow \mathbb{R}$ is convex on $[u_k, v_k]$ for each $k = 1, 2, \dots, n$. From Hermite-Hadamard inequality, we have almost everywhere*

$$\begin{aligned}
&\sum_{k=1}^{n-1} X\left(\left(t_1, \dots, t_{k-1}, \frac{2u_k v_k}{v_k + u_k}, \frac{2u_{k+1} v_{k+1}}{v_{k+1} + u_{k+1}}, \dots, t_n\right), \cdot\right) \\
&\leq \sum_{k=1}^{n-1} \frac{u_k v_k}{v_k - u_k} \int_{u_k}^{v_k} \frac{X_{t_n}^k\left(\frac{2u_{k+1} v_{k+1}}{u_{k+1} + v_{k+1}}, \cdot\right)}{t_k^2} dt_k \\
&\leq \sum_{k=1}^{n-1} \frac{u_k v_k u_{k+1} v_{k+1}}{(v_k - u_k)(v_{k+1} - u_{k+1})} \int_{u_k}^{v_k} \int_{u_{k+1}}^{v_{k+1}} \frac{X_{t_n}^{k+1}(t_{k+1}, \cdot)}{(t_k t_{k+1})^2} dt_{k+1} dt_k
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{n-1} \frac{u_k v_k}{v_k - u_k} \int_{u_k}^{v_k} \frac{X_{t_n}^k(u_{k+1}, \cdot) + X_{t_n}^k(v_{k+1}, \cdot)}{2t_k^2} dt_k \\
&\leq \frac{1}{4} \sum_{k=1}^{n-1} \left[X((t_1, \dots, t_{k-1}, u_k, u_{k+1}, \dots, t_n), \cdot) + X((t_1, \dots, t_{k-1}, v_k, u_{k+1}, \dots, t_n), \cdot) \right. \\
&\quad \left. + X((t_1, \dots, t_{k-1}, u_k, v_{k+1}, \dots, t_n), \cdot) + X((t_1, \dots, t_{k-1}, v_k, v_{k+1}, \dots, t_n), \cdot) \right]. \quad (3.2)
\end{aligned}$$

Proof. Since $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ is harmonically convex stochastic on n -coordinates, there for $X_{t_n}^i: [u_i, v_i] \times \Omega \rightarrow \mathbb{R}$ is harmonically convex stochastic process on $[u_i, v_i]$ for each $i = 1, 2, \dots, n$, we have on $[u_{i+1}, v_{i+1}]$ almost everywhere

$$X_{t_n}^{i+1}\left(\frac{2u_{i+1}v_{i+1}}{u_{i+1} + v_{i+1}}, \cdot\right) \leq \frac{u_{i+1}v_{i+1}}{v_{i+1} - u_{i+1}} \int_{u_{i+1}}^{v_{i+1}} \frac{X_{t_n}^{i+1}(t_{i+1}, \cdot)}{t_{i+1}^2} dt_{i+1} \leq \frac{X_{t_n}^{i+1}(u_{i+1}, \cdot) + X_{t_n}^{i+1}(v_{i+1}, \cdot)}{2}.$$

All of sides of the above inequalities by integrating over $[u_i, v_i]$

$$\begin{aligned}
&\frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}\left(\frac{2u_{i+1}v_{i+1}}{u_{i+1} + v_{i+1}}, \cdot\right)}{t_i^2} dt_i \\
&\leq \frac{u_i v_i u_{i+1} v_{i+1}}{(v_i - u_i)(v_{i+1} - u_{i+1})} \int_{u_i}^{v_i} \int_{u_{i+1}}^{v_{i+1}} \frac{X_{t_n}^{i+1}(t_{i+1}, \cdot)}{t_{i+1}^2} dt_{i+1} dt_i \\
&\leq \frac{u_i v_i}{2(v_i - u_i)} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(u_{i+1}, \cdot) + X_{t_n}^{i+1}(v_{i+1}, \cdot)}{t_i^2} dt_i. \quad (3.3)
\end{aligned}$$

Again applying the Hermite-Hadamard inequality

$$X\left(\left(t_1, \dots, \frac{2u_i v_i}{v_i + u_i}, \frac{2u_{i+1} v_{i+1}}{u_{i+1} + v_{i+1}}, \dots, t_n\right), \cdot\right) \leq \frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}\left(\frac{2u_{i+1} v_{i+1}}{u_{i+1} + v_{i+1}}, \cdot\right)}{t_i^2} dt_i \quad (3.4)$$

for each $i \in \{1, 2, \dots, n-1\}$ and also

$$\begin{aligned}
&\frac{u_i v_i}{2(v_i - u_i)} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(u_{i+1}, \cdot) + X_{t_n}^{i+1}(v_{i+1}, \cdot)}{t_i^2} dt_i \\
&= \frac{1}{2} \left[\frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(u_{i+1}, \cdot)}{t_i^2} dt_i + \frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(v_{i+1}, \cdot)}{t_i^2} dt_i \right] \\
&\leq \frac{1}{2} \left[\frac{X((t_1, \dots, t_{i-1}, u_i, u_{i+1}, \dots, t_n), \cdot) + X((t_1, \dots, t_{i-1}, v_i, u_{i+1}, \dots, t_n), \cdot)}{2} \right. \\
&\quad \left. + \frac{X((t_1, \dots, t_{i-1}, u_i, v_{i+1}, \dots, t_n), \cdot) + X((t_1, \dots, t_{i-1}, v_i, v_{i+1}, \dots, t_n), \cdot)}{2} \right] \\
&= \frac{1}{4} \left[X((t_1, \dots, t_{i-1}, u_i, u_{i+1}, \dots, t_n), \cdot) + X((t_1, \dots, t_{i-1}, v_i, u_{i+1}, \dots, t_n), \cdot) \right. \\
&\quad \left. + X((t_1, \dots, t_{i-1}, u_i, v_{i+1}, \dots, t_n), \cdot) + X((t_1, \dots, t_{i-1}, v_i, v_{i+1}, \dots, t_n), \cdot) \right] \quad (3.5)
\end{aligned}$$

for each $i \in \{1, 2, \dots, n-1\}$. Using the inequalities (3.4) and (3.5) in (3.3) and taking summation from 1 to $n-1$, we have (3.2).

Theorem 3.2. Let $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ be harmonically convex stochastic process on n -coordinates. Then we have almost everywhere

$$\begin{aligned} & \sum_{k=1}^n \frac{u_k v_k}{v_k - u_k} \int_{u_k}^{v_k} (X_{u_n}^k(t_k, \cdot) + X_{v_n}^k(t_k, \cdot)) dt_k \\ & \leq \frac{n}{2} [X(\mathbf{u}, \cdot) + X(\mathbf{v}, \cdot)] + \frac{1}{2} \sum_{k=1}^n [X_{u_n}^k(v_k, \cdot) + X_{v_n}^k(u_k, \cdot)] \end{aligned}$$

where $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$.

Proof. Since $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ is harmonically convex stochastic on n -coordinates, there for $X_{t_n}^i: [u_i, v_i] \times \Omega \rightarrow \mathbb{R}$ is harmonically convex stochastic process on $[a_i, b_i]$ for each $i = 1, 2, \dots, n$, we have almost everywhere

$$\begin{aligned} \frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} X_{u_n}^i(t_i, \cdot) dt_i &= \frac{X_{u_n}^i(u_i, \cdot) + X_{u_n}^i(v_i, \cdot)}{2} \leq \frac{X(\mathbf{u}, \cdot) + X_{u_n}^i(v_i, \cdot)}{2} \\ \frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} X_{v_n}^i(t_i, \cdot) dt_i &= \frac{X_{v_n}^i(u_i, \cdot) + X_{v_n}^i(v_i, \cdot)}{2} \leq \frac{X_{v_n}^i(u_i, \cdot) + X(\mathbf{v}, \cdot)}{2} \end{aligned}$$

Adding above two inequalities we have

$$\frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} [X_{u_n}^i(t_i, \cdot) + X_{v_n}^i(t_i, \cdot)] dt_i \leq \frac{X(\mathbf{u}, \cdot) + X(\mathbf{v}, \cdot) + X_{u_n}^i(v_i, \cdot) + X_{v_n}^i(u_i, \cdot)}{2},$$

where $i = 1, 2, \dots, n$. Adding above n inequalities we get the desired results of this theorem.

Corollary 3.1. Let $X: \Delta^2 \subset \mathbb{R}_+^2 \times \Omega \rightarrow \mathbb{R}$ be harmonically convex stochastic process on 2-coordinates. Then (2.3) is valid.

Theorem 3.3. Let $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \rightarrow \mathbb{R}$ be harmonically convex stochastic process on n -coordinates. Then we have almost everywhere

$$\begin{aligned} & X\left(\left(\frac{2u_1 v_1}{v_1 + u_1}, \dots, \frac{2u_{n-1} v_{n-1}}{v_{n-1} + u_{n-1}}, \frac{2u_n v_n}{v_n + u_n}\right), \cdot\right) \\ & \leq \left(\prod_{i=1}^n \frac{u_i v_i}{v_i - u_i}\right) \int_{u_1}^{v_1} \dots \int_{u_n}^{v_n} \frac{X_{t_n}^n(t_n, \cdot)}{(\prod_{i=1}^n t_i)^2} dt_n \dots dt_1 \leq \frac{1}{2^n} \sum_{\delta \in l_i(n)} X(\delta u + (1 - \delta)v, \cdot), \quad (3.6) \end{aligned}$$

where $l_i(n) := \{\delta \in \mathbb{N}_0^n: \delta \leq 1, |\delta| = n + 1 - i, i = 1, \dots, n + 1\}$, ; $|\delta| := \delta_1 + \dots + \delta_n \in \mathbb{N}$; $\delta u := (\delta_1 u_1, \dots, \delta_n u_n) \in \mathbb{N}_0^n$ for $u, v \in \Delta^n$.

Proof. By applying Hermite-Hadamard's inequality for harmonically convex stochastic process $X_{t_n}^n$ on interval $[u_n, v_n]$ we have almost everywhere

$$X_{t_n}^n\left(\frac{2u_n v_n}{u_n + v_n}, \cdot\right) \leq \frac{u_n v_n}{v_n - u_n} \int_{u_n}^{v_n} \frac{X_{t_n}^n(t_n, \cdot)}{t_n^2} dt_n \leq \frac{X_{t_n}^n(u_n, \cdot) + X_{t_n}^n(v_n, \cdot)}{2}.$$

By integrating on $[u_{n-1}, v_{n-1}]$, we get

$$\begin{aligned}
& \frac{u_{n-1}v_{n-1}}{v_{n-1}-u_{n-1}} \int_{u_{n-1}}^{v_{n-1}} \frac{X_{t_n}^n \left(\frac{2u_n v_n}{u_n + v_n}, \cdot \right)}{t_{n-1}^2} dt_{n-1} \\
& \leq \frac{u_{n-1}v_{n-1}u_n v_n}{(v_{n-1}-u_{n-1})(v_n-u_n)} \int_{u_{n-1}}^{v_{n-1}} \int_{u_n}^{v_n} \frac{X_{t_n}^n(t_n, \cdot)}{(t_{n-1}t_n)^2} dt_{n-1} dt_n \\
& \leq \frac{u_{n-1}v_{n-1}}{v_{n-1}-u_{n-1}} \int_{u_{n-1}}^{v_{n-1}} \frac{X_{t_n}^n(u_n, \cdot) + X_{t_n}^n(v_n, \cdot)}{2t_{n-1}^2} dt_{n-1}. \tag{3.7}
\end{aligned}$$

From (3.4), (3.5), respectively

$$X \left(\left(t_1, \dots, \frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}}, \frac{2u_n v_n}{u_n + v_n} \right), \cdot \right) \leq \frac{u_{n-1}v_{n-1}}{v_{n-1}-u_{n-1}} \int_{u_{n-1}}^{v_{n-1}} \frac{X_{t_n}^n \left(\frac{2u_n v_n}{u_n + v_n}, \cdot \right)}{t_{n-1}^2} dt_{n-1}, \tag{3.8}$$

$$\begin{aligned}
& \frac{u_{n-1}v_{n-1}}{2(v_{n-1}-u_{n-1})} \int_{u_{n-1}}^{v_{n-1}} \frac{X_{t_n}^n(u_n, \cdot) + X_{t_n}^n(v_n, \cdot)}{t_{n-1}^2} dt_{n-1} \\
& \leq \frac{1}{2^2} \left[X((t_1, \dots, u_{n-1}, u_n), \cdot) + X((t_1, \dots, v_{n-1}, u_n), \cdot) \right] \\
& \quad + X((t_1, \dots, u_{n-1}, v_n), \cdot) + X((t_1, \dots, v_{n-1}, v_n), \cdot). \tag{3.9}
\end{aligned}$$

From (3.7)-(3.9)

$$\begin{aligned}
& X \left(\left(t_1, \dots, \frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}}, \frac{2u_n v_n}{u_n + v_n} \right), \cdot \right) \\
& \leq \frac{u_{n-1}v_{n-1}u_n v_n}{(v_{n-1}-u_{n-1})(v_n-u_n)} \int_{u_{n-1}}^{v_{n-1}} \int_{u_n}^{v_n} \frac{X_{t_n}^n(t_n, \cdot)}{(t_{n-1}t_n)^2} dt_{n-1} dt_n \\
& \leq \frac{1}{2^2} \left[X((t_1, \dots, u_{n-1}, u_n), \cdot) + X((t_1, \dots, v_{n-1}, u_n), \cdot) \right] \\
& \quad + X((t_1, \dots, u_{n-1}, v_n), \cdot) + X((t_1, \dots, v_{n-1}, v_n), \cdot). \tag{3.10}
\end{aligned}$$

Thus, integrating on $[u_{n-2}, v_{n-2}]$, we have

$$\begin{aligned}
& \frac{u_{n-2}v_{n-2}}{v_{n-2}-u_{n-2}} \int_{u_{n-2}}^{v_{n-2}} \frac{X \left(\left(t_1, \dots, \frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}}, \frac{2u_n v_n}{u_n + v_n} \right), \cdot \right)}{t_{n-2}^2} dt_{n-2} \\
& \leq \left(\prod_{i=n-2}^n \frac{u_i v_i}{v_i - u_i} \right) \int_{u_{n-2}}^{v_{n-2}} \int_{u_{n-1}}^{v_{n-1}} \int_{u_n}^{v_n} \frac{X_{t_n}^n(t_n, \cdot)}{(\prod_{i=n-2}^n t_i)^2} dt_{n-2} dt_{n-1} dt_n \\
& \leq \frac{1}{2^2} \frac{u_{n-2}v_{n-2}}{v_{n-2}-u_{n-2}} \int_{u_{n-2}}^{v_{n-2}} \frac{1}{t_{n-2}^2} \left[X((t_1, \dots, u_{n-1}, u_n), \cdot) + X((t_1, \dots, v_{n-1}, u_n), \cdot) \right] \\
& \quad + X((t_1, \dots, u_{n-1}, v_n), \cdot) + X((t_1, \dots, v_{n-1}, v_n), \cdot) \Big] dt_{n-2}. \tag{3.11}
\end{aligned}$$

From (3.4), (3.5), respectively

$$X \left(\left(t_1, \dots, \frac{2u_{n-2}v_{n-2}}{v_{n-2}+u_{n-2}}, \frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}}, \frac{2u_n v_n}{u_n + v_n} \right), \cdot \right)$$

$$\leq \frac{u_{n-2}v_{n-2}}{v_{n-2} - u_{n-2}} \int_{u_{n-2}}^{v_{n-2}} \frac{X\left((t_1, \dots, \frac{2u_{n-1}v_{n-1}}{v_{n-1} + u_{n-1}}, \frac{2u_nv_n}{u_n + v_n}), \cdot\right)}{t_{n-2}^2} dt_{n-2}, \quad (3.12)$$

$$\begin{aligned} &\leq \frac{1}{2^2} \frac{u_{n-2}v_{n-2}}{v_{n-2} - u_{n-2}} \int_{u_{n-2}}^{v_{n-2}} \frac{1}{t_{n-2}^2} \left[X((t_1, \dots, u_{n-1}, u_n), \cdot) + X((t_1, \dots, v_{n-1}, u_n), \cdot) \right. \\ &\quad \left. + X((t_1, \dots, u_{n-1}, v_n), \cdot) + X((t_1, \dots, v_{n-1}, v_n), \cdot) \right] dt_{n-2} \\ &\leq \frac{1}{2^3} \left[\begin{aligned} &X((t_1, \dots, u_{n-2}, u_{n-1}, u_n), \cdot) + X((t_1, \dots, v_{n-2}, u_{n-1}, u_n), \cdot) \\ &+ X((t_1, \dots, u_{n-2}, v_{n-1}, u_n), \cdot) + X((t_1, \dots, v_{n-2}, v_{n-1}, u_n), \cdot) \\ &+ X((t_1, \dots, u_{n-2}, u_{n-1}, v_n), \cdot) + X((t_1, \dots, v_{n-2}, u_{n-1}, v_n), \cdot) \\ &+ X((t_1, \dots, u_{n-2}, v_{n-1}, v_n), \cdot) + X((t_1, \dots, v_{n-2}, v_{n-1}, v_n), \cdot) \end{aligned} \right]. \end{aligned} \quad (3.13)$$

From (3.11)-(3.13)

$$\begin{aligned} &\frac{u_{n-2}v_{n-2}}{v_{n-2} - u_{n-2}} \int_{u_{n-2}}^{v_{n-2}} \frac{X\left((t_1, \dots, \frac{2u_{n-1}v_{n-1}}{v_{n-1} + u_{n-1}}, \frac{2u_nv_n}{u_n + v_n}), \cdot\right)}{t_{n-2}^2} dt_{n-2} \\ &\leq \left(\prod_{i=n-2}^n \frac{u_i v_i}{v_i - u_i} \right) \int_{u_{n-2}}^{v_{n-2}} \int_{u_{n-1}}^{v_{n-1}} \int_{u_n}^{v_n} \frac{X_{t_n}^n(t_n, \cdot)}{(\prod_{i=n-2}^n t_i)^2} dt_{n-2} dt_{n-1} dt_n \\ &\leq \frac{1}{2^3} \left[\begin{aligned} &X((t_1, \dots, u_{n-2}, u_{n-1}, u_n), \cdot) + X((t_1, \dots, u_{n-2}, v_{n-1}, u_n), \cdot) \\ &+ X((t_1, \dots, u_{n-2}, u_{n-1}, v_n), \cdot) + X((t_1, \dots, u_{n-2}, v_{n-1}, v_n), \cdot) \\ &+ X((t_1, \dots, v_{n-2}, u_{n-1}, u_n), \cdot) + X((t_1, \dots, v_{n-2}, v_{n-1}, u_n), \cdot) \\ &+ X((t_1, \dots, v_{n-2}, u_{n-1}, v_n), \cdot) + X((t_1, \dots, v_{n-2}, v_{n-1}, v_n), \cdot) \end{aligned} \right]. \end{aligned} \quad (3.14)$$

Using inductive method for $n = k - 1$, we obtain

$$\begin{aligned} &X\left(\left(\frac{2u_1v_1}{v_1 + u_1}, \dots, \frac{2u_{k-2}v_{k-2}}{v_{k-2} + u_{k-2}}, \frac{2u_{k-1}v_{k-1}}{v_{k-1} + u_{k-1}}\right), \cdot\right) \\ &\leq \left(\prod_{i=1}^{k-1} \frac{u_i v_i}{v_i - u_i} \right) \int_{u_1}^{v_1} \dots \int_{u_{k-1}}^{v_{k-1}} \frac{X_{t_{k-1}}^{k-1}(t_{k-1}, \cdot)}{(\prod_{i=1}^{k-1} t_i)^2} dt_n \dots dt_1 \leq \frac{1}{2^{k-1}} \sum_{\delta \in l_i(k-1)} X(\delta u + (1 - \delta)v, \cdot) \end{aligned}$$

where $l_i(k-1) := \{\delta \in \mathbb{N}_0^n : \delta \leq 1, |\delta| = k - i; i = 1, \dots, k, k \in \mathbb{N}\}$, $|\delta| := \delta_1 + \dots + \delta_n \in \mathbb{N}$; $\delta u := (\delta_1 u_1, \dots, \delta_n u_n) \in \mathbb{N}_0^n$ for $u, v \in \Delta^n$. Consequently, for $n = k$, we get (3.6).

Example 1. Let $X: \Delta^3 \times \Omega \rightarrow \mathbb{R}$ be a harmonically convex stochastic process and be integrated in mean-square on Δ^3 . Then almost everywhere

$$\begin{aligned} &X\left(\left(\frac{2u_1v_1}{v_1 + u_1}, \frac{2u_2v_2}{v_2 + u_2}, \frac{2u_3v_3}{v_3 + u_3}\right), \cdot\right) \\ &\leq \frac{u_1v_1u_2v_2u_3v_3}{(v_1 - u_1)(v_2 - u_2)(v_3 - u_3)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} \int_{u_3}^{v_3} \frac{X((t_1, t_2, t_3), \cdot)}{t_1^2 t_2^2 t_3^2} dt_3 dt_2 dt_1 \end{aligned}$$

$$\leq \frac{1}{2^3} \begin{bmatrix} X((u_1, u_2, u_3), \cdot) + X((v_1, u_2, u_3), \cdot) \\ + X((u_1, v_2, u_3), \cdot) + X((v_1, v_2, u_3), \cdot) \\ + X((u_1, u_2, v_3), \cdot) + X((v_1, u_2, v_3), \cdot) \\ + X((u_1, v_2, v_3), \cdot) + X((v_1, v_2, v_3), \cdot) \end{bmatrix}$$

Proof. According to Theorem 2 for $n = 3$, we get $X_{t_3}^3(t_3, \cdot) := X((t_1, t_2, t_3), \cdot)$ and $l_i(3) := \{\delta \in \mathbb{N}_0^3: \delta \leq 1, |\delta| = 4 - i\}, i = 1, 2, 3, 4$. Then

$$l_1(3) = \{(1, 1, 1)\}; \quad l_2(3) = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\},$$

$$l_3(3) = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}; \quad l_4(3) = \{(0, 0, 0)\},$$

and for $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \Delta^3$

$$\begin{aligned} & \sum_{\delta \in l_1(3)} X(\delta u + (1 - \delta)v, \cdot) \\ &= X((1, 1, 1)(u_1, u_2, u_3) + [(1, 1, 1) - (1, 1, 1)](v_1, v_2, v_3), \cdot) = X((u_1, u_2, u_3), \cdot); \\ & \sum_{\delta \in l_2(3)} X(\delta u + (1 - \delta)v, \cdot) = X((0, 1, 1)(u_1, u_2, u_3) + [(1, 1, 1) - (0, 1, 1)](v_1, v_2, v_3), \cdot) \\ & \quad + X((1, 0, 1)(u_1, u_2, u_3) + [(1, 1, 1) - (1, 0, 1)](v_1, v_2, v_3), \cdot) \\ & \quad + X((1, 1, 0)(u_1, u_2, u_3) + [(1, 1, 1) - (1, 1, 0)](v_1, v_2, v_3), \cdot) \\ &= X((u_1, v_2, v_3), \cdot) + X((u_1, v_2, u_3), \cdot) + X((u_1, u_2, v_3), \cdot); \end{aligned}$$

So,

$$\sum_{\delta \in l_3(3)} X(\delta u + (1 - \delta)v, \cdot) = X((v_1, v_2, u_3), \cdot) + X((v_1, u_2, v_3), \cdot) + X((u_1, v_2, v_3), \cdot);$$

$$\sum_{\delta \in l_4(3)} X(\delta u + (1 - \delta)v, \cdot) = X((v_1, v_2, v_3), \cdot).$$

Thus

$$\begin{aligned} & \sum_{\delta \in l_i(3)} X(\delta u + (1 - \delta)v, \cdot) = X((u_1, u_2, u_3), \cdot) \\ & \quad + X((u_1, v_2, v_3), \cdot) + X((u_1, v_2, u_3), \cdot) + X((u_1, u_2, v_3), \cdot) \\ & \quad + X((v_1, v_2, u_3), \cdot) + X((v_1, u_2, v_3), \cdot) + X((u_1, v_2, v_3), \cdot) + X((v_1, v_2, v_3), \cdot). \end{aligned}$$

Using all of the above equalities in (3.6), we obtain the desired result in this example.

4. CONCLUSION

In this paper, we obtain some new Hermite-Hadamard type inequalities for harmonically convex stochastic processes on n -coordinates and obtain. As special cases, one can obtain several new and correct versions of the previously known results for various classes of these stochastic processes.

Acknowledgements

The author would like to thank to Assoc. Prof. İmdat İŞCAN, Giresun University, Department of Mathematics, owing to his ideas.

References

- [1] Dragomir, S.S., Pearce, C.E., Sandor, M.J., A note on the Jensen-Hadamard inequality, *Anal. Num. Theor. Approx.*, 1990, 19, 29-34.
- [2] De la Cal, J., Carcamo, J., Multidimensional Hermite-Hadamard inequalities and the convex order, *J. Math. Anal. Appl.*, 2006, 324, 248-261.
- [3] Alomari, M. W., A generalization of Hermite-Hadamard's inequality, *Kragujevac J. Math.*, 2017, 41(1.2), 313-328.
- [4] Mwaeze, E. R., Generalized Hermite-Hadamard's inequality for functions convex on the coordinates, *Applied Mathematics E-Notes*, 2018, 18, 275-283.
- [5] Ellahi, H., Farid, G. and Rehman, A. U., Hadamard's Inequality for s -convex function on n -coordinates, *Proceedings of 1st ICAM Attock, Pakistan*, 2015.
- [6] Nikodem, K., On convex stochastic processes, *Aequat. Math.*, 1980, 20, 184-197.
- [7] Shaked, M., Shanthikumar, J.G., Stochastic convexity and its applications, *Advances in Applied Probability*, 1988, 20, 427-446.
- [8] Skowronski, A., On some properties of J -convex stochastic processes, *Aequat. Math.*, 1992, 44, 249-258.
- [9] Kotrys, D., Hermite-Hadamard inequality for convex stochastic processes, *Aequat. Math.*, 2012, 83, 143-151.
- [10] Set, E., Sarıkaya, M.Z., Tomar, M., Hermite-Hadamard inequalities for co-ordinates convex stochastic processes, *Mathematica Aeterna*, 2015, 5 (1.2), 363-382.
- [11] Okur, N., İşcan, İ., Yüksek Dizdar, E., Hermite-Hadamard inequalities for harmonically convex stochastic processes, *International Journal of Economic and Administrative Studies*, 2018, 11 (18. EYI Special Issue), 281-292.
- [12] Okur, N., İşcan, İ., Usta, Y., Some Integral inequalities for harmonically convex stochastic processes on the coordinates, *Advanced Math. Models & Applications*, 2018, 3(1), 63-75.
- [13] Karahan, V., Okur, N., Hermite-Hadamard Type Inequalities for Convex Stochastic Processes On n -Coordinates, Special Issue: Proceedings of International Conference on Mathematicd and Mathematics Education (ICMME 2018), *Turk. J. Math. Comput. Sci.*, 10 (2018) 256-262.
- [14] Vilorio, J.M., Cortez, M.V., Hermite-Hadamard type inequalities for harmonically convex functions on n -coordinates, *Appl. Math. Inf. Sci. Lett.*, 2018, 6(2), 1-6.